Exercise 84

Draw a diagram showing two perpendicular lines that intersect on the y-axis and are both tangent to the parabola $y = x^2$. Where do these lines intersect?

Solution

Consider two lines in the xy-plane.

$$y = m_1 x + b_1$$

$$y = m_2 x + b_2$$

If the two lines are perpendicular, then

$$m_1 = m \quad \text{and} \quad m_2 = -\frac{1}{m}.$$

And if these two lines have the same y-intercept, then

$$\text{At } x = 0: \quad m_1 x + b_1 = m_2 x + b_2 \quad \rightarrow \quad b_1 = b_2 = b.$$

Consequently, the two lines are

$$y = mx + b$$

$$y = -\frac{1}{m}x + b.$$

Take the derivative of the parabola's equation.

$$y' = \frac{d}{dx}(x^2) = 2x$$

Find the value of x that the first line is tangent to the parabola.

$$m = 2x \quad \rightarrow \quad x = \frac{m}{2}$$

Find the value of x that the second line is tangent to the parabola.

$$-\frac{1}{m} = 2x \quad \rightarrow \quad x = -\frac{1}{2m}$$

Plug these values of x into $y = x^2$ to find the corresponding y-values on the parabola.

$$y = \left(\frac{m}{2}\right)^2 = \frac{m^2}{4}$$

$$y = \left(-\frac{1}{2m}\right)^2 = \frac{1}{4m^2}$$

Plug these corresponding x- and y-values into the equations for the lines to determine m and b.

$$\left(\frac{m^2}{4}\right) = m\left(\frac{m}{2}\right) + b$$

$$\left(\frac{1}{4m^2}\right) = -\frac{1}{m}\left(-\frac{1}{2m}\right) + b$$

Solving the system yields

$$m = -1 \qquad b = -\frac{1}{4}.$$

Therefore, the two lines are

$$y = -x - \frac{1}{4}$$

$$y = x - \frac{1}{4},$$

and they intersect on the y-axis at $(0, -1/4)$. Below is a graph of these lines together with the parabola.